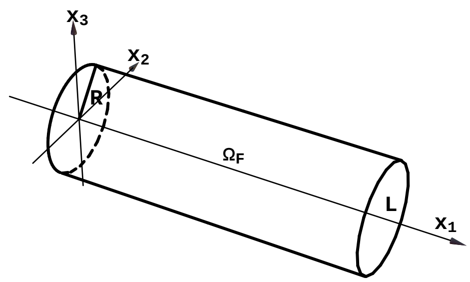
Womersley flow

Description of the problem

We consider the fluid flow through a straight pipe of length l and radius R. Fluid is assumed to be incompressible with density ϱ and viscosity μ. Unknowns in the problem are velocity of the fluid $\boldsymbol{v}$ and pressure p. The coordinate system in $\mathbb{R}^3$ we choose as in the figure: Points in $\mathbb{R}^3$ we denote by



$x = (x_1, x_2, x_3)$. The flow is governed by prescribing the pressure at both pipe's ends, i.e.

$$p(x,t)|_{x_1=0} = p_0(t), \quad p(x,t)|_{x_1=l} = p_L(t), \tag{1}$$

for given time dependant functions p_0 and p_L; on the sequel we assume that they are $2T-$ periodic. There ia an additional assumption in (1): the pressure at pipe's ends is independent of the space variable, i.e. constant on each pipe's ends. This assumption allows important simplification of the problem. Still, the usage of the boundary condition as in (1) in hemodynamics is unclear since measuring pressure on two cross-sections in an non-invasive way is questionable. For instance, using color doppler, 2-d velocity profile can be obtained but with questionable accuracy. This kind of fluid flow problems was considered by a British mathematician John Ronald Womersley, see `https://en.wikipedia.org/wiki/John_R._Womersley`.

Since the forcing that drives the flow is $2T-$periodic we can look for the $2T-$periodic solution given by the velocity $\boldsymbol{v}$ and pressure p. Mathematical formulation of the problem is then given by:

find $2T-$periodic functions $\boldsymbol{v}$ and p such that

$$\left.\begin{array}{l} \varrho \dfrac{\partial \boldsymbol{v}}{\partial t} - \mu \Delta \boldsymbol{v} + \operatorname{grad} p = 0 \quad \text{u cijevi, } t > 0, \\ \operatorname{div} \boldsymbol{v} = 0 \quad \text{u cijevi, } t > 0 \text{ (inkompresibilnost)}, \\ \boldsymbol{v} = 0 \quad \text{na plaštu cijevi, } t > 0 \text{ (viskoznost fluida)}, \\ p(x,t)|_{x_1=0} = p_0(t), \quad p(x,t)|_{x_1=l} = p_L(t), \ t > 0. \end{array}\right\} \tag{2}$$

- The equation $(2)_1$ is nonstationary (i.e., evolution) Stokes equation; in this special case of the flow through a pipe (pressure at the pipe's ends is constant in the space, the pipe is straight and with the constant cross–section) the same analysis can be applied at the nonlinear analogue, i.e., the Navier-Stokes equation (in $(2)_1$ the nonlinear transport term $(\boldsymbol{v}\operatorname{grad})\boldsymbol{v}$ is added).

- The system (2) gives an approximation of the blood flow in straight parts of arteries; Womersley, in 1955., see [3], found a closed formula fo the solution of (2) for particular p_0 and p_L.
- Lately, the problem as in (2) was explored by Heywood, Rannacher i Turek (1996) and Veneziani (2000). In both papers the boundary condition of the type $(2)_4$ is discussed and justified; furthermore Veneziani shows numerically existence of counterflow for sinusoidal pressure drop $p_0 - p_L$.

In the sequel we repeat and extend this topic.

Simplification of the problem

It can be shown that the problem (2) is well posed in variational (weak) sense, see [1] or [2]; the solution $(\boldsymbol{v}, p)$ exists and is unique (the pressure is unique up to a constant, as usual). The condition of $2T-$periodicity of the solution replaces prescription of the initial condition for the velocity.

We try with the following ansatz:

$$\left.\begin{array}{l} \boldsymbol{v}(x_1,x_2,x_3,t) = u(x_2,x_3,t)\,\mathbf{e}_1 \;\; \text{time–dependent Poisseuille flow,} \\ p(x_1,x_2,x_3,t) = \dfrac{p_l(t)-p_0(t)}{l}\,x_1 + p_0(t), \\ u \;\; 2T - \text{periodic skalar function.} \end{array}\right\} \tag{3}$$

The flow with the velocity profile as in $(2)_1$ is laminar; every such is also incompressible, i.e., the equation $(2)_2$ is fulfilled as the last two equations of the system $(2)_1$. The boundary condition $(2)_4$ is fulfilled by the pressure of the form $(3)_2$. From (2), and from the Navier-Stokes system, we obtain the following parabolic problem:

$$\left.\begin{array}{l} \varrho\,\dfrac{\partial u}{\partial t} - \mu\,\Delta_{2,3}u + \dfrac{p_l - p_0}{l} = 0, \quad x_2^2 + x_3^2 < R^2, \; t > 0, \\ u \;\; 2T - \text{periodic function,} \\ u = 0 \;\; \text{za } x_2^2 + x_3^2 = R^2, \; t > 0. \end{array}\right\} \tag{4}$$

Here we use the notation $\Delta_{2,3} = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

Let

$$a(t) = \frac{p_l(t) - p_0(t)}{l}$$

and introduce the polar coordinates (r, φ). Since there is no explicit dependance on φ in (4), we assume that (without changing the notation for the velocity function)

$$u(x_2, x_3, t) = u(r, t).$$

As in polar coordinates we have

$$\Delta = \frac{1}{r}\,\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\,\frac{\partial^2}{\partial \varphi^2},$$

from (4) we obtain the problem

$$\left.\begin{array}{l} \varrho\,\dfrac{\partial u}{\partial t} - \mu\dfrac{1}{r}\,\dfrac{\partial}{\partial r}(r\dfrac{\partial}{\partial r})u + a = 0, \quad r \in (0,R), \; t > 0, \\ u \;\; 2T - \text{periodic function,} \\ u(R,t) = 0, \; t > 0. \end{array}\right\} \tag{5}$$

By the assumption a is $2T-$periodic function. Such functions can be expanded in the classical Fourier series on $[0, 2T]$; note that the convergence in the series will depend on the smoothness of a. For the moment let us assume that

$$a(t) = \alpha \cos \omega t + \beta \sin \omega t, \tag{6}$$

where $\alpha,\ \beta \in \mathbf{R},\ \alpha^2 + \beta^2 > 0$, are given and

$$\omega = \frac{k\,\pi}{T} \ \text{ za neki } k \in \{0, 1, \ldots\}.$$

For such functions a the solution of the problem (4) can be written in a closed form using complex Bessel functions.

However if (6) does not hold, for example if a is general $2T-$periodic function, a possible numerical approximation of (4) can be built as in the following three steps.

1. step. Approximate a by $n-$th partial sum its Fourier series,

$$S_n(a) = \frac{a_0(a)}{2} + \sum_{k=1}^{n} (a_k(a) \cos \omega_k t + b_k(a) \sin \omega_k t);$$

$\omega_k = k\pi/T$, $a_k(a)$ and $b_k(a)$ are the Fourier coefficients of the function a, i.e.,

$$a_k(a) = \frac{1}{T} \int_0^{2T} a(t) \cos \omega_k t \, dt, \quad k = 0, 1, \ldots,$$

$$b_k(a) = \frac{1}{T} \int_0^{2T} a(t) \sin \omega_k t \, dt, \quad k = 1, 2, \ldots,$$

2. step. For $k \in \{0, 1, \ldots, n\}$ we solve the problem (4) with a_k of the form (6) instead of a,

$$a_k(t) = a_k(a) \cos \omega_k t + b_k(a) \sin \omega_k t;$$

its solution we denote by u_k.

3. step. Define the function $U_n = u_0 + \ldots + u_n$ which is, by the linearity of the problem (4), reasonable approximation of the solution of (4).

The form of the function a from (6) suggest the following form of the solution of (4):

$$u(r, t) = c(r) \cos \omega t + s(r) \sin \omega t, \tag{7}$$

where c and s are unknown functions. Inserting (7) in $(5)_1$ we obtain the following system for c and s:

$$-\mu \frac{d}{dr}(r\, c'(r)) - \varrho\omega\, s(r) = -\beta,$$

$$-\mu \frac{d}{dr}(r\, s'(r)) + \varrho\omega\, c(r) = -\alpha,$$

The condition of $2T-$periodicity is fulfilled because of (7), and boundary condition $(5)_3$ is reduced to

$$c(R) = s(R) = 0.$$

Since the above problem is singular boundary value problem on $(0, R)$ with regular singularity in $r = 0$, we add the boundedness of the solution as the boundary condition in $r = 0$; in this case this boundary condition is effectively expressed by

$$c'(0) = s'(0) = 0.$$

Let

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{X}(r) = \begin{bmatrix} s(r) \\ c(r) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\beta \\ -\alpha \end{bmatrix}.$$

System for functions c and s is then given by:

$$\left.\begin{aligned} -\mu\,[r\,\mathbf{X}'(r)]' + r\omega\varrho\,\mathbf{J}\,\mathbf{X}(r) = r\,\mathbf{B}, \quad r \in (0, R), \\ \mathbf{X}'(0) = 0, \quad \mathbf{X}(R) = 0. \end{aligned}\right\} \tag{8}$$

Womersley (1955) solved similar system by a closed formula. Let

$$a(t) = Re(A\exp(i\omega t));$$

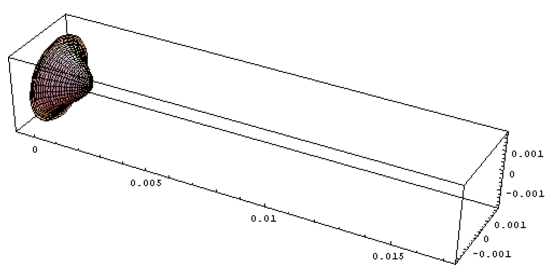
where A is given complex number, i imaginary unit, Re denotes real part of the complex number and ω is as before, see (6). Let $\nu = \mu/\varrho$ (kinematical viscosity). Then the function

$$Re\left\{\frac{A}{\rho}\,\frac{1}{i\omega}\left[1 - \frac{J_0(ri^{\frac{3}{2}}\sqrt{\frac{\omega}{\nu}})}{J_0(Ri^{\frac{3}{2}}\sqrt{\frac{\omega}{\nu}})}\right]\exp(i\omega t)\right\}$$

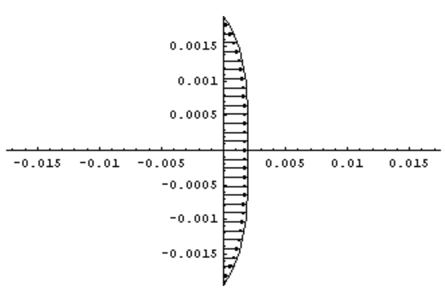
is a solution of the problem (5) (the Womersley solution). Here J_0 is Bessel function of zeroth order with complex argument.

Visualization of the solution

As in the poiseuille case we again conclude that the velocity is equal on all cross–sections, which implies that it is enough to visualize the solution on one cross–section. However, here the velocity is time t dependent. Therefore we need animation to fully describe the solution. One screenshot is given by The



assumed axial symmetry the solution on all cross–sections depends only on the distance to the center of the cross–section. Therefore it is enough to present the solution on one diameter. The velocity profile here



connects all vectors.

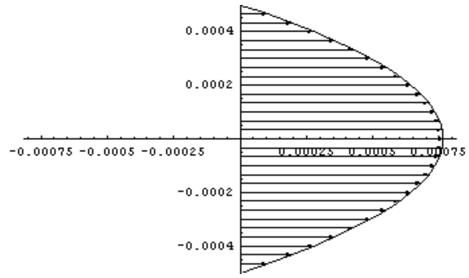


Figure 1: small radius, profile close to parabolic

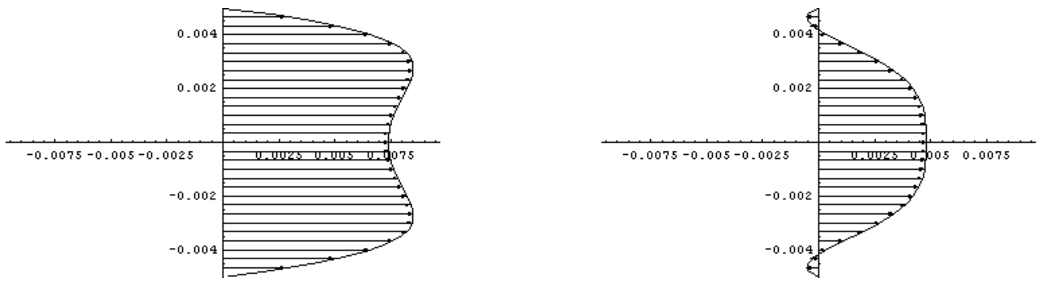
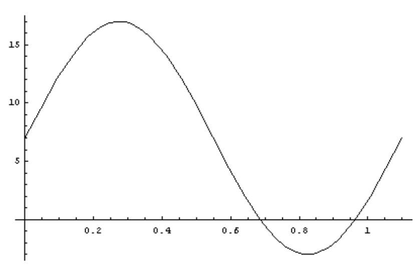


Figure 2: larger radius, counterflow observed

Different types of the solution can be seen on the following screenshots of Womersley solutions.
For the prescribed pressures

$$p_0(t) = 7 + 10 \sin \omega t, \qquad p_L(t) = 0$$

we obtain a solution of the Navier-Stokes system as a sum of two terms, the Poiseuille velocity and one



Womersley solution (for ω). The solutions for different radiuses are given on the following figures We

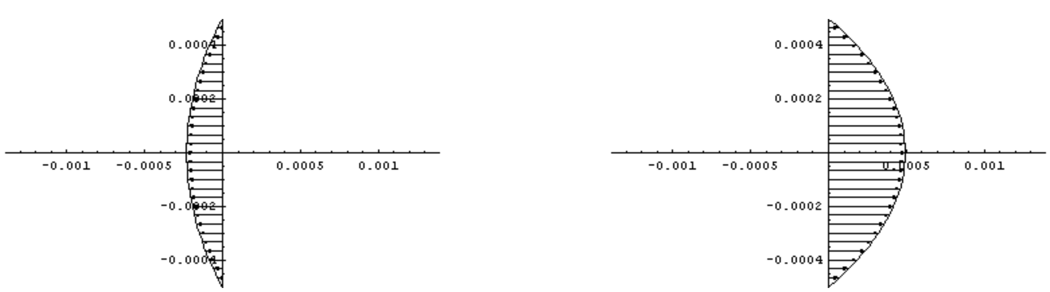


Figure 3: small radius

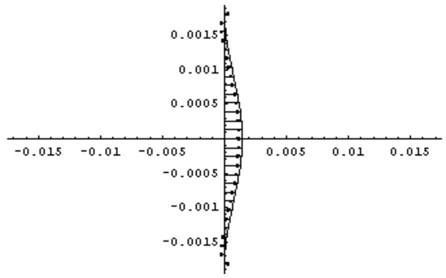


Figure 4: middle radius

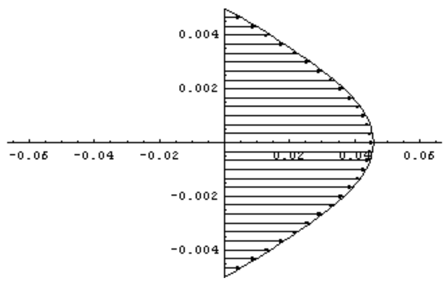


Figure 5: large radius

can note that there are times at which the pressure is negative, but the average pressure is positive. This implies that overall flow will be directed from entrance ($z = 0$) to exit ($z = L$). The velocity is shown to be significantly radius dependant. If the radius is small at some times the flow is globally in opposite direction. However if the radius is large enough the flow is directed in the same direction at all times, while for some radiuses in between these cases there appears only local counterflow. Similar effect can be seen with respect to the viscosity of the fluid.

References

[1] Heywood J., Rannacher R., Turek S., Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations, Int. J. Num. Meth. Fl. 22, 325-352, 1996.

[2] Veneziani A., Boundary conditions for blood flow problems, preprint, 2000.

[3] J.R. Womersley, *Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known*, J. Physiol. 127, 553-563, 1955.