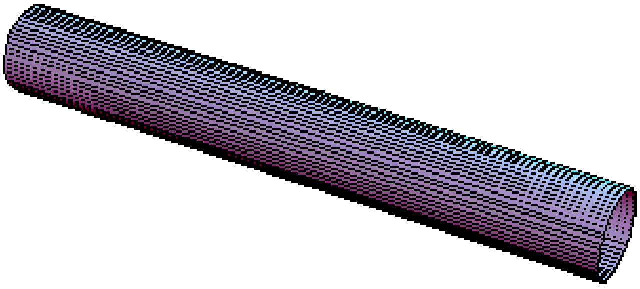
Poiseuilleov flow

We consider stationary flow (velocity in each point is not changing in time) of incompressible viscous fluid through the straight pipe of length L, with constant circular cross–section of radius R, with prescribed



constant (independent of time and space) pressures p_0 and p_L at the entrance and exit of the pipe. This kind of the flow was explored by Poiseuille (see `http://scienceworld.wolfram.com/biography/Poiseuille.html`). Mathematical formulation of the problem is given by

$$
\begin{aligned}
&\varrho_F \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\, \boldsymbol{v} \right) - \mu_F \Delta \boldsymbol{v} + \nabla p = 0, \\
&\operatorname{div} \boldsymbol{v} = 0, \\
&\boldsymbol{v}|_{r=R} = 0, \\
&p|_{z=0} = p_0, \quad v_r|_{z=0} = 0, \\
&p|_{z=L} = p_L, \quad v_r|_{z=L} = 0.
\end{aligned}
$$

We look for axially–symmetric solutions, so rewrite the problem i cylindrical coordinates

$$
\begin{aligned}
&\varrho_F \left(\frac{\partial v_r}{\partial r} v_r + \frac{\partial v_r}{\partial z} v_z \right) - \mu_F \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{1}{r^2} v_r + \frac{\partial^2 v_r}{\partial z^2} \right) + \frac{\partial p}{\partial r} = 0, \\
&\varrho_F \left(\frac{\partial v_z}{\partial r} v_r + \frac{\partial v_z}{\partial z} v_z \right) - \mu_F \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) + \frac{\partial p}{\partial z} = 0, \\
&\frac{\partial v_r}{\partial r} + \frac{1}{r} v_r + \frac{\partial v_z}{\partial z} = 0.
\end{aligned}
$$

Furthermore we assume there is a solution such that $v_r = 0$, so the problem is further simplified

$$
\begin{aligned}
&\frac{\partial p}{\partial r} = 0, \\
&\varrho_F \left(\frac{\partial v_z}{\partial z} v_z \right) - \mu_F \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) + \frac{\partial p}{\partial z} = 0, \\
&\frac{\partial v_z}{\partial z} = 0.
\end{aligned}
$$

First we conclude

$$p(z), \quad v_z(r),$$

and then p and v_z satisfy the equation

$$-\mu_F \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + r\frac{\partial p}{\partial z} = 0.$$

Integrating over r we obtain

$$r\frac{\partial v_z}{\partial r} = \frac{1}{\mu_F}\frac{r^2}{2}\frac{\partial p}{\partial z}.$$

Therefore

$$\frac{\partial v_z}{\partial r} = \frac{1}{\mu_F}\frac{r}{2}\frac{\partial p}{\partial z},$$

so

$$v_z(r) = \frac{R^2 - r^2}{4\mu_F}\frac{\partial p(z)}{\partial z},$$

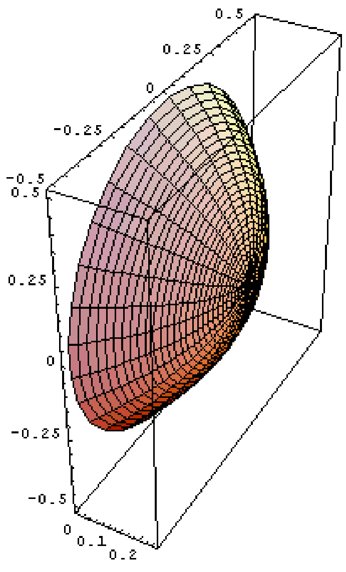
Since v_z is independent of z we conclude that the pressure p is linear function of z. Since the values of the pressure are prescribed at the pipe's ends we obtain

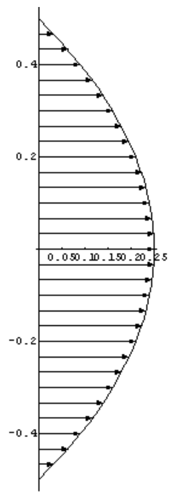
$$p(z) = \frac{p_L - p_0}{L} z + p_0.$$

Thus we have found one stationary solution of the incompressible Navier-Stokes system for constant pressure ate the pipe ends:

$$v_z(r) = \frac{p_L - p_0}{L}\frac{1}{4\mu_F}\left(R^2 - r^2\right), \qquad p(z) = \frac{p_L - p_0}{L} z + p_0.$$

We conclude that the velocity is constant on all cross–sections, which implies that it is enough to present it on a single cross–section. In this example the cross–section is circular with radius R Velocity profile is then given by By the assumed axial symmetry of the solution it depends only on the distance of the point from





the center of the circle. Thus the solution is ususally shown just on one diameter (radius). Interpretation of the velocity profile is then to connect all velocity vectors as in the figure..