Effective model of a fluid flow through an elastic tube

1 Introduction

This note is motivated by the study of blood flow in compliant arteries. In medium to large vessels blood is usually modelled as an incompressible Newtonian fluid. The vessel walls we consider to be thin, behaving as a prestressed linearly elastic membrane shell. The flow is driven by the inlet and outlet pressure that is periodic in time. We derive a closed, effective model that approximates the 3D problem to the ε^2–accuracy, where ε is the aspect ratio of the problem, defined in (1.1). For details see [1], [2], [3].

We first formulate the three-dimensional problem. Then we derive the a priori estimates from the energy equality of the problem. These estimates are used to define nondimensional quantities in the problem. This enables us to compare the terms in the problem and to build a simplified approximation. For straight tubes there are two basic assumptions:

$$\varepsilon = \frac{\max R}{L} \tag{1.1}$$

is small (the undeformed radius is small compared to the length of the tube) and the deflection of the tube wall is small compared to the radius. We obtain the equations that are closed and of mixed, hyperbolic–parabolic type with memory. The memory effects capture the viscoelastic nature of the coupled fluid–structure interaction problem.

2 Setting up the problem

We study the flow of an incompressible Newtonian fluid in an axially symmetric cylindrical domain with variable radius governed by the time-dependent inlet and outlet boundary data. In the reference configuration the length of the domain is denoted by L. For a given smooth function $R : [0, L] \to \mathbb{R}$, the radius of the cylinder at $z \in [0, L]$ is denoted by $R(z)$. The reference domain is now defined by (see Figure 1)

$$\Omega = \left\{ (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 : r \in (0, R(z)), \theta \in (0, 2\pi), z \in (0, L) \right\}$$

and its lateral boundary is given by

$$\Sigma = \left\{ (R(z)\cos\theta, R(z)\sin\theta, z) \in \mathbb{R}^3 : \theta \in (0, 2\pi), z \in (0, L) \right\}.$$

We assume that the domain is thin and long, i.e. the nondimensional pa-

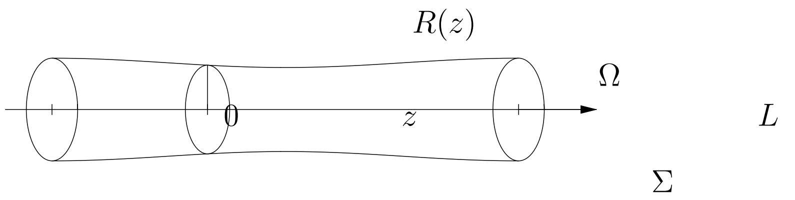

Figure 1: The reference domain

rameter $\varepsilon = \frac{R_{\max}}{L}$ is small, where $R_{\max} = \max_{z \in [0,L]} R(z)$.

The lateral boundary is assumed to be elastic and to deform as a result of the fluid–structure interaction between the fluid and the structure. To be precise we assume that Σ behaves as a homogeneous, isotropic, linearly elastic membrane shell with thickness h and that at the reference configuration the shell is prestressed by $T^0_{\theta\theta} = p_{\text{ref}} \frac{R_{\max}}{h}$, where $T^0_{\theta\theta}$ is the θ, θ component of the stress tensor. Moreover, we account for the radial displacements η of the shell only. Therefore the strain, given by the linear part of the change of metric tensor of the shell, is given by

$$\mathbf{G}(\eta) = \frac{1}{2} \mathrm{Lin} \begin{pmatrix} (R' + \eta')^2 - (R')^2 & 0 \\ 0 & (R + \eta)^2 - R^2 \end{pmatrix} = \begin{pmatrix} R'\eta' & 0 \\ 0 & R\eta \end{pmatrix}.$$

Here $'$ denotes the derivative with respect to the longitudinal variable. Then, for a given radial component of the force f_r, the model equation for the boundary behavior in the weak formulation is given by

$$\int_0^L h\rho_S \frac{\partial^2 \eta}{\partial t^2} \xi R \sqrt{1 + (R')^2} dz + \int_0^L \Big(\frac{\sigma h E}{1 - \sigma^2} \left(\frac{R'}{1 + (R')^2} \eta' + \frac{1}{R} \eta \right) \left(\frac{R'}{1 + (R')^2} \xi' + \frac{1}{R} \xi \right)$$

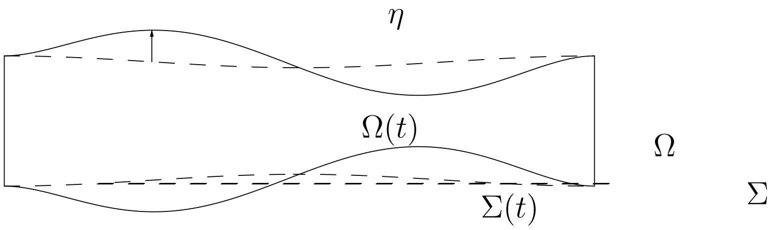

Figure 2: The deformed domain

$$+\frac{hE}{1+\sigma}\left(\left(\frac{R'}{1+(R')^2}\right)^2\eta'\xi'+\frac{1}{R^2}\eta\xi\right)\Big)R\sqrt{1+(R')^2}dz \tag{2.1}$$
$$+\int_0^L hT^0_{\theta\theta}\frac{\eta\xi}{R^2}R\sqrt{1+(R')^2}dz=\int_0^L f_r\xi R\sqrt{1+(R')^2}dz, \qquad \xi\in H^1_0(0,L);$$

here E is the Young modulus, σ the Poisson ratio, ρ_S is the shell density. For simplicity we only take the membrane part of the Koiter shell model. Note, as well, that the radial component η is not the component of the displacement normal to the shell.

With these assumptions on the geometry of the problem, the moving domain $\Omega(t)$ at time t is given by

$$\Omega(t)=\left\{(r\cos\theta,r\sin\theta,z)\in\mathbb{R}^3: r\in(0,R(z)+\eta(t,z)),\theta\in(0,2\pi),z\in(0,L)\right\},$$

while the wall of the cylinder at time t is described by

$$\Sigma(t)=\left\{((R(z)+\eta(t,z))\cos\theta,(R(z)+\eta(t,z))\sin\theta,z)\in\mathbb{R}^3:\theta\in(0,2\pi),z\in(0,L)\right\},$$

see Figure 2. We also denote the inlet and outlet boundary $B_0(t)=\partial\Omega(t)\cap\{z=0\}, B_L(t)=\partial\Omega(t)\cap\{z=L\}$.

Now we search for the axially symmetric solution (v_r,v_z,η), where $\boldsymbol{v}=(v_r,v_z)$ is the fluid velocity, of the problem defined by the following:

a) the fluid satisfies the incompressible Navier–Stokes equations in $\Omega(t)$

$$\rho_F\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}\right)-\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}\right)+\frac{\partial p}{\partial r}=0,$$
$$\rho_F\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)-\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right)+\frac{\partial p}{\partial z}=0,$$
$$\frac{\partial v_r}{\partial r}+\frac{\partial v_z}{\partial z}+\frac{v_r}{r}=0.$$

Here p is the pressure, μ is the fluid dynamic viscosity coefficient and ρ_F is the fluid density;

b) the moving boundary $\Sigma(t)$ behaves as the linearly membrane shell defined by the equations (2.1);

c) the kinematic condition on the contact $\Sigma(t)$ of the fluid and the structure is the continuity of the velocity

$$v_r(R+\eta(z,t),z,t) = \frac{\partial \eta(z,t)}{\partial t}, \quad v_z(R+\eta(z,t),z,t) = 0;$$

d) the dynamic condition on the contact $\Sigma(t)$ of the fluid and the structure is the continuity of the contact force. Since the radial component of the fluid contact force $[(p-p_{\text{ref}})\mathbf{I} - 2\mu D(\boldsymbol{v})]\,\boldsymbol{n}\cdot\boldsymbol{e}_r$ is given in the Eulerian coordinates, where p_{ref} is the reference pressure, and the structure contact force (2.1) is given in the Lagrangian coordinates, we must take into account the Jacobian of the transformation from the Eulerian to the Lagrangian coordinates $J := \sqrt{\det((\nabla\boldsymbol{\phi})^T\nabla\boldsymbol{\phi})} = \sqrt{(R+\eta)^2\,(1+(R'+\eta')^2)}$, where $\boldsymbol{\phi} : (0,L)\times[0,2\pi) \to \mathbb{R}^3$ and its gradient $\nabla\boldsymbol{\phi}$ are defined by

$$\begin{array}{l} x = (R+\eta)\cos\theta \\ y = (R+\eta)\sin\theta \\ z = z \end{array}, \quad \nabla\phi = \begin{pmatrix} (R'+\eta')\cos\theta & -(R+\eta)\sin\theta \\ (R'+\eta')\sin\theta & (R+\eta)\cos\theta \\ 1 & 0 \end{pmatrix}.$$

Here $\boldsymbol{n}$ is the unit normal at the deformed structure

$$\boldsymbol{n} = -\frac{R'+\eta'}{\sqrt{1+(R'+\eta')^2}}\boldsymbol{e}_z + \frac{1}{\sqrt{1+(R'+\eta')^2}}\boldsymbol{e}_r.$$

The coupling is then performed by requiring that for every Borel subset B of the lateral boundary Σ, the contact force exerted by the fluid to the structure equals, but is of opposite sign to the contact force exerted by the structure to the fluid, namely,

$$\int_B [(p-p_{\text{ref}})\mathbf{I} - 2\mu D(\boldsymbol{v})]\,\boldsymbol{n}\cdot\boldsymbol{e}_r J d\theta dz = \int_B f_r R\sqrt{1+(R')^2}d\theta dz$$

and so, pointwise, the dynamic coupling condition reads

$$[(p-p_{\text{ref}})\mathbf{I} - 2\mu D(\boldsymbol{v})]\,(-(R'+\eta')\boldsymbol{e}_z+\boldsymbol{e}_r)\cdot\boldsymbol{e}_r\left(1+\frac{\eta}{R}\right)\frac{1}{\sqrt{1+(R')^2}} = f_r,$$

where f_r is given in (2.1).

e) boundary conditions (inlet/outlet conditions)

$$\begin{aligned} &v_r = 0, \;\; p + \rho_F (v_z)^2/2 = P_0(t) + p_{\text{ref}} \quad \text{on} \quad B_0(t), \\ &v_r = 0, \; p + \rho_F (v_z)^2/2 = P_L(t) + p_{\text{ref}} \quad \text{on} \quad B_L(t), \\ &\eta = 0 \;\; \text{for} \;\; z = 0, \quad \eta = 0 \;\; \text{for} \;\; z = L \;\; \text{and} \; \forall t \in \mathbb{R}_+, \end{aligned}$$

for P_0, P_L given functions of t only.

f) initial conditions

$$\eta = \frac{\partial \eta}{\partial t} = 0 \quad \text{and} \quad \boldsymbol{v} = 0 \quad \text{on} \quad \Sigma \times \{0\}.$$

The problem defined by a)–f) is a three–dimensional fluid–structure interaction problem. It is the starting point for our analysis.

3 The a priori estimates

The main step in the derivation of the effective equations approximating the problem a)–f) are the a priori estimates. These estimates provide the magnitudes of the unknown functions in the problem $\boldsymbol{v}, p, \eta$ which we use to write the nondimensional equations.

Now we rescale time by introducing the nondimensional time $\bar{t} = \omega t$, where ω is the characteristic frequency, which is specified later on such that the ballanced estimted is obtained

$$\omega = \frac{1}{L}\sqrt{\frac{Q_{\min}}{2\rho_F}};$$

then ωL, in the constant cross–section case ($R = R_{\max}$), is exactly the same as the structure "sound speed" obtained by Fung. To simplify notation we keep t to denote the rescaled time. We also denote

$$Q = \left(T^0_{\theta\theta} + \frac{E}{1+\sigma}\right)\frac{h}{R}, \quad Q_{\min} = \min Q \quad R_{\min} = \min R, \quad R_{\max} = \max R,$$

and

$$\hat{p}(t) = \frac{A(t)}{L} z + P_0(t), \qquad A(t) = P_L(t) - P_0(t).$$

Theorem 3.1 *The solution* $(\boldsymbol{v}, \eta)$ *of the fluid-structure problem satisfies the estimate*

$$\frac{\varrho_F}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega(t))} + 2\mu\|D(\boldsymbol{v})\|^2_{L^2(\Omega)} + \pi\omega^2\rho_S h R_{\min}\left\|\frac{\partial\eta}{\partial t}\right\|^2 + \frac{\pi}{2}Q_{\min}\|\eta\|^2 \;\le\; \frac{16\pi L R^2_{\max}}{Q_{\min}}\mathcal{P}^2,$$

where $\mathcal{P}^2 := \sup_{z,t}|\hat{p}|^2 + \left(\sup_z \int_0^t |\partial_t \hat{p}| d\tau\right)^2 + T\int_0^t |A(\tau)|^2$.

Corollary 3.1 *The solution* $(\boldsymbol{v}, \eta)$ *of the fluid-structure problem satisfies the estimates*

$$\frac{1}{R^2_{\max}\pi L}\|\boldsymbol{v}\|^2 \le \frac{32}{\varrho_F Q_{\min}}\mathcal{P}^2, \qquad \frac{1}{L}\|\eta\|^2 \le \frac{32R^2_{\max}}{Q^2_{\min}}\mathcal{P}^2.$$

4 The reduced problem

Since the a priori estimates obtained from Theorem 3.1 present the upper bounds for the behavior of the unknown functions we use the scaled upper bounds to only capture how the magnitude of the unknown functions changes with a given parameter. Using these estimates we are able to derive the nondimensional problem and to detect the small terms. Denote

$$\varepsilon = \frac{R_{\max}}{L}, \qquad \delta = \frac{\Xi}{R_{\max}}, \qquad \gamma = \frac{h}{R_{\max}}.$$

Then we write the nondimensional equations. In these equations we can detect small terms and neglect ε^2 terms to obtain the ε^2 approximation. We obtain the approximation in the form

$$v_z = v_z^{0,0} + v_z^{0,1} + v_z^{1,0}, \qquad v_r = v_r^{1,0}, \quad \eta = \eta^{0,0} + \eta^{0,1} + \eta^{1,0}, \quad p = p^{0,0} + p^{0,1},$$

where the 0th order model is given by

$$\begin{aligned}
&\frac{\partial \eta^{0,0}}{\partial t} + \frac{1}{R}\frac{\partial}{\partial z}\int_0^R r v_z^{0,0} dr = 0,\\
&\varrho_F\frac{\partial v_z^{0,0}}{\partial t} - \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{0,0}}{\partial r}\right) = -\frac{\partial}{\partial z}\left(\left(\frac{E}{1-\nu^2} + T^0_{\theta\theta}\right)\frac{h}{R^2}\eta^{0,0}\right), \quad (4.1)\\
&v_z^{0,0}|_{r=0} - \text{ bounded}, \quad v_z^{0,0}|_{t=R} = 0, \quad v_z^{0,0}|_{t=0} = 0,\\
&\eta^{0,0}|_{t=0} = 0, \quad \eta^{0,0}|_{z=0} = P_0/C, \quad \eta^{0,0}|_{z=L} = P_L/C,
\end{aligned}$$

where

$$C = \left(\frac{E}{1-\nu^2} + T^0_{\theta\theta}\right)\frac{h}{R^2} \tag{4.2}$$

The δ correction

$$\begin{aligned}
&\frac{\partial \eta^{0,1}}{\partial t} + \frac{1}{R}\frac{\partial}{\partial z}\int_0^R r v_z^{0,1} dr = -\frac{1}{2R}\frac{\partial}{\partial t}\left(\eta^{0,0}\right)^2, \\
&\varrho_F \frac{\partial v_z^{0,1}}{\partial t} - \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{0,1}}{\partial r}\right) = -\frac{\partial}{\partial z}\left(\left(\frac{E}{1-\nu^2} + T^0_{\theta\theta}\right)\frac{h}{R^2}\left(\eta^{0,1} - \frac{(\eta^{0,0})^2}{R}\right)\right) \qquad (4.3)\\
&v_z^{0,1}|_{r=0} - \text{ bounded}, \quad v_z^{0,1}|_{r=R} = -\eta^{0,0}\frac{\partial v_z^{0,0}}{\partial r}|_{r=R}, \quad v_z^{0,1}|_{t=0} = 0, \\
&\eta^{0,1}|_{t=0} = 0, \quad \eta^{0,1}|_{z=0} = 0, \quad \eta^{0,1}|_{z=L} = 0.
\end{aligned}$$

The ε correction

$$v_r^{1,0}(r,z,t) = \frac{1}{r}\left(R\frac{\partial \eta^{0,0}}{\partial t} + \int_r^R \xi \frac{\partial v_z^{0,0}}{\partial z}(\xi, z, t) d\xi\right), \tag{4.4}$$

$$\rho_F \frac{\partial v_z^{1,0}}{\partial t} - \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{1,0}}{\partial r}\right) = -\rho_F\left(v_r^{1,0}\frac{\partial v_z^{0,0}}{\partial r} + v_z^{0,0}\frac{\partial v_z^{0,0}}{\partial z}\right), \tag{4.5}$$

$$v_z^{1,0}|_{\bar{r}=0} - \text{ bounded}, \qquad v_z^{1,0}|_{r=R} = 0, \qquad v_z^{1,0}|_{t=0} = 0.$$

Viscoelastic nature of the model

The probem (4.1) can be solved by considering the auxiliary problem

$$\begin{aligned}
&\frac{\partial \zeta}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \zeta}{\partial r}\right) = 0 \quad \text{in } (0,R)\times(0,\infty) \\
&\zeta|_{r=0} \text{ is bounded }, \quad \zeta|_{R=0} = 0 \text{ and } \zeta|_{t=0} = 1,
\end{aligned}$$

and the mean of ζ in the radial direction $\mathcal{K}(t) = 2\int_0^R \zeta(r,t)\ r dr$, which can both be evaluated in terms of the Bessel's functions. Our solution can then be written in terms of the following operator $(\mathcal{K} \star f)(z,t) := \int_0^t \mathcal{K}(\frac{\mu(t-\tau)}{\rho_F}) f(z,\tau) d\tau$. Now the problem for $\eta^{0,0}$ expressed in terms of

$$p^{0,0} = C\eta^{0,0}$$

consists of finding $p^{0,0}$ by solving the following initial-boundary value problem of the Biot type with memory:

$$\begin{aligned}
&\frac{\partial p^{0,0}}{\partial t}(z,t) = \frac{C}{2\rho_F R}\frac{\partial^2(\mathcal{K}\star p^{0,0})}{\partial z^2}(z,t) \text{ on } (0,L)\times(0,+\infty) \\
&p^{0,0}|_{z=0} = P_0, \quad p^{0,0}|_{z=L} = P_L \quad \text{and} \quad p^{0,0}|_{t=0} = 0.
\end{aligned}$$

This approach uncovers the visco-elastic nature of the coupled fluid-structure interaction problem since the resulting equations have the form of a Biot system with memory.

5 Numerical Method

First rewrite the (4.1) approximation in terms of $v_z^{0,0}$ and $p^{0,0}$. Multiply the first equation in (4.1) by C (see (4.2)) and take the derivative with respect to t and then substitute $\frac{\partial v_z^{0,0}}{\partial t}$ from the second equation. Therefore instead of (4.1), we solve the hyperbolic-parabolic system

$$\frac{\partial^2 p^{0,0}}{\partial t^2} - \frac{CR}{2\rho_F}\frac{\partial^2 p^{0,0}}{\partial z^2} = -\frac{C\mu}{\rho_F}\frac{\partial}{\partial z}\left(\frac{\partial v_z^{0,0}}{\partial r}|_{r=R}\right), \tag{5.1}$$

$$\rho_F \frac{\partial v_z^{0,0}}{\partial t} - \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{0,0}}{\partial r}\right) = -\frac{\partial p^{0,0}}{\partial z}, \tag{5.2}$$

with the initial and boundary conditions

$$v_z^{0,0}|_{r=0} - \text{ bounded}, \quad v_z^{0,0}|_{t=R} = 0, \quad v_z^{0,0}|_{t=0} = 0,$$
$$p^{0,0}|_{t=0} = 0, \quad p^{0,0}|_{z=0} = P_0, \quad p^{0,0}|_{z=L} = P_L.$$

Perform the same computation for the $0,1$ approximation and replace (4.3) by

$$\frac{\partial^2 p^{0,1}}{\partial t^2} - \frac{CR}{2\rho_F}\frac{\partial^2 p^{0,1}}{\partial z^2} = -\frac{C\mu}{\rho_F}\frac{\partial}{\partial z}\left(\frac{\partial v_z^{0,0}}{\partial r}|_{r=R}\right) - \frac{CR}{2\rho_F}\frac{\partial^2}{\partial z^2}\left(\frac{C}{R}(\eta^{0,0})^2\right)$$
$$-\frac{C}{2R}\frac{\partial^2}{\partial t^2}\left(\eta^{0,0}\right)^2, \tag{5.3}$$

$$\rho_F \frac{\partial v_z^{0,1}}{\partial t} - \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{0,1}}{\partial r}\right) = -\frac{\partial}{\partial z}\left(p^{0,1} - \frac{C}{R}(\eta^{0,0})^2\right), \tag{5.4}$$

with the initial and boundary conditions given by

$$v_z^{0,1}|_{r=0} - \text{ bounded}, \quad v_z^{0,1}|_{r=R} = -\eta^{0,0}\frac{\partial v_z^{0,0}}{\partial r}|_{r=R}, \quad v_z^{0,1}|_{t=0} = 0,$$
$$p^{0,1}|_{t=0} = 0, \quad p^{0,1}|_{z=0} = 0, \quad p^{0,1}|_{z=L} = 0.$$

The problems for $(p^{0,0}, v_z^{0,0})$ and $(p^{0,1}, v_z^{0,1})$ are of the same form. It is a coupled system of the hyperbolic (for the displacement of the domain wall) and parabolic equation (for the velocity). The velocity equations are given on the cross–sections $r \in (0, R(z))$ of the domain with no explicit dependence on each other with longitudinal variable z as a parameter. The dependence is incorporated through the wave equation for the displacement of the wall.

The approximation $1, 0$ is straightforward once the approximations $0, 0$ and $0, 1$ are obtained. The systems for the $0, 0$ and $0, 1$ approximations have the same form, with the mass and stiffness matrices equal for both problems, depending only on the variable z through the cross–section $(0, R(z))$. Thus they are generated only once for each cross–section.

The problems $0, 0$ and $0, 1$ are solved simultaneously using a time-iteration procedure. First solve the parabolic equation for $v_z^{0,0}$ at the time step t_{i+1} by explicitly evaluating the right–hand side at the time-step t_i. Then solve the wave equation for $\eta^{0,0}$ with the evaluation of the right–hand side at the time-step t_{i+1}. Using these results for $v_z^{0,0}$ and $\eta^{0,0}$, computed at t_{i+1}, obtain a correction at t_{i+1} by repeating the process with the updated values of the right–hand sides. An implicit time discretization is used. The numerical algorithm reads:

1. Approximation $0, 0$: for $i = 0$ to n_T
   (a) solve (5.2) at t_{i+1} for $v_z^{0,0}$ using 1D FEM with linear elements
   (b) solve (5.1) at t_{i+1} for $p^{0,0}$ using 1D FEM with C^1 elements
   (c) compute $\eta^{0,0}$

2. Approximation $0, 1$: for $i = 0$ to n_T
   (a) solve (5.4) at t_{i+1} for $v_z^{0,1}$ using 1D FEM with linear elements
   (b) solve (5.3) at t_{i+1} for $p^{0,1}$ using 1D FEM with C^1 elements
   (c) compute $\eta^{0,1}$

3. Approximation $1, 0$
   (a) solve (4.4) for $v_r^{1,0}$ using numerical integration
   (b) solve (4.5) for $v_z^{1,0}$ using 1D FEM with linear elements

4. Compute the total approximation $v_r = v_r^{1,0}, v_z = v_z^{0,0} + v_z^{0,1} + v_z^{1,0}, \eta = \eta^{0,0} + \eta^{0,1}$.

In this algorithm a sequence of 1D problems is solved, so the numerical complexity is that of 1D solvers.

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